ALGORITHMS & ADVANCED DATA STRUCTURES (#3)

O-NOTATION

ADAPTED FROM
CS 146 SJSU (KATERINA POTIKA)

Measures of Algorithm Complexity

- **Worst-Case Running Time**: the longest time for any input size of **n**
  - an upper bound on running time for any input

- **Best-Case Running Time**: the shortest time for any input size of **n**
  - an lower bound on running time for any input

- **Average-Case Behavior**: the expected performance averaged over all possible inputs
  - it is generally better than worst case behavior, but sometimes it's roughly as bad as worst case

Order of Growth

- For very large input size n, it is the *rate of grow*, or *order of growth* that matters asymptotically
  - ignore the *lower-order terms*, since they are relatively insignificant for very large *n*
  - ignore *leading term's constant coefficients*, since they are not as important for the rate of growth in computational efficiency for very large *n*

- **Higher order** functions of *n* are normally considered **less efficient**

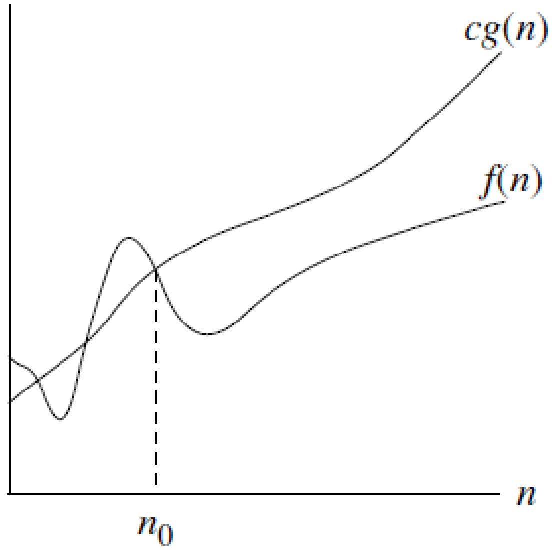
O - Notation

FORMAL DEFINITIONS

Big-O notation
(Upper Bound – Worst Case)

O-notation

$O(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Big-O notation
(Upper Bound – Worst Case)

- A mathematically formal way of ignoring constant factors, and looking only at the “shape” of the function

- $f(n)=O(g(n))$ should be considered as saying that “$f(n)$ is at most $g(n)$, up to constant factors”.

- We usually will have $f(n)$ be the running time of an algorithm and $g(n)$ a nicely written function
- *Example*: The running time of insertion sort algorithm is $O(n^2)$

Big-O notation examples

Example: $2n^2 = O(n^3)$, with $c = 1$ and $n_0 = 2$.

Examples of functions in $O(n^2)$:

n^2
$n^2 + n$
$n^2 + 1000n$
$1000n^2 + 1000n$
Also,
n
$n/1000$
$n^{1.99999}$
$n^2/\lg\lg\lg n$

Big-O notation
(Upper Bound – Worst Case)

- ignore the multiplicative constants and the lower order terms, e.g.,
  - *n, n+1, n+80, 40n, n + log n* *is O(?)*
  - $n^{1.1} + \mathbf{10000000000n}$ is *O(?)*
  - n^2 is *O(?)*
  - $\mathbf{3}n^2 + \mathbf{6n + log\ n + 24.5}$ is *O(?)*

Practice

- ***What is the O-notation of $f(N)=3n^2 + 6n + \log n + 24.5$***

A. O(n)

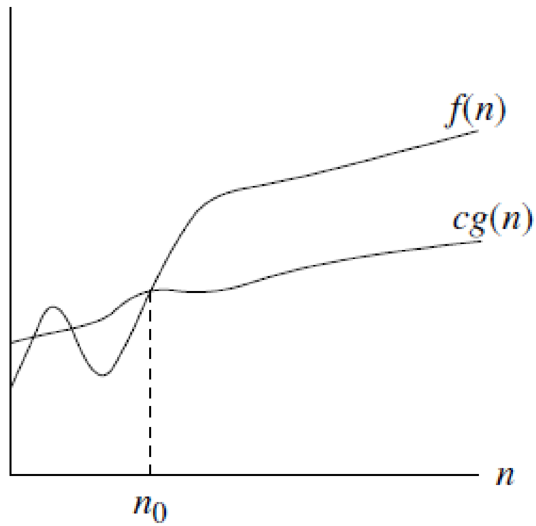
B. $O(n^2)$

C. $O(n^3)$

D. B and C

Ω-notation (Omega)
(Lower Bound – Best Case)

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}\ .$$



$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Ω-notation (Omega)

- We say Insertion Sort's run time $T(n)$ is $\Omega(n)$
  - Why?
- For example
  - the worst-case running time of insertion sort is $O(n^2)$, and
  - the best-case running time of insertion sort is $\Omega(n)$

Ω-notation (Omega)

Example: $\sqrt{n} = \Omega(\lg n)$, with $c = 1$ and $n_0 = 16$.

Examples of functions in $\Omega(n^2)$:

n^2

$n^2 + n$

$n^2 - n$

$1000n^2 + 1000n$

$1000n^2 - 1000n$

Also,

n^3

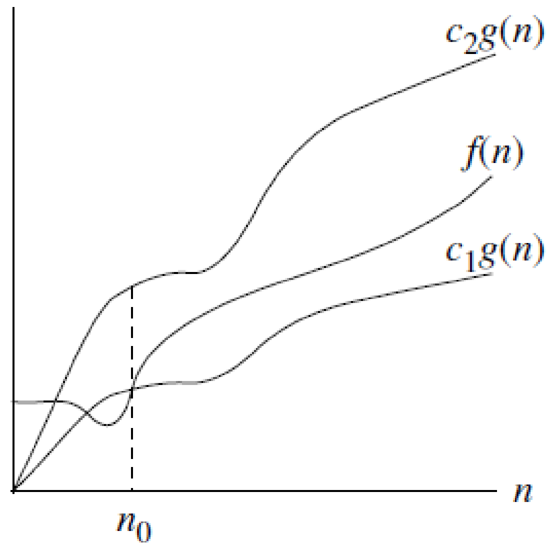
$n^{2.00001}$

$n^2 \lg \lg \lg n$

2^{2^n}

Θ notation (Theta) (Tight Bound)

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Θ notation (Theta)

- We say *g(n)* is an *asymptotic tight bound* for *f(n)*:

- **Theta notation**
  - **Θ(g(n))** means that as **n** → ∞, the execution time **f(n)** is at **most** $\mathbf{c_2.g(n)}$ and at **least** $\mathbf{c_1.g(n)}$ for some constants $\mathbf{c_1}$ and $\mathbf{c_2}$.

- f(n) = Θ(g(n)) if and only if
  - f(n) = O(g(n)) & f(n) = Ω(g(n))

Θ notation (Theta) - Example

- **<u>Example1:</u>**
- Show that $6n^3 \neq \Theta(n^2)$
- Suppose for the purpose of contradiction that $\mathbf{c_2}$ and $\mathbf{n_o}$ exist such that $6n^3 \leq c_2 n^2$ for all $n \geq n_0$
  - Dividing by n^2 yields
    - $n \leq c_2/6$
  - which cannot possibly hold for arbitrary large **n**, since $\mathbf{c_2}$ is constant
  - Also, $lim_{n\to\infty}[6n^3 / n^2] = lim_{n\to\infty}[6n] = \infty$, *which is not a non-zero constant*

o-notation

We say $g(n)$ is an *upper bound* for $f(n)$ that is *not* asymptotically tight (strictly).

$$o(g(n)) = \{f(n) : \text{ for all constants } c > 0 \text{, there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\}$$

Another view, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$

$n^{1.9999} = o(n^2)$

$n^2/\lg n = o(n^2)$

$n^2 \ne o(n^2)$ (just like $2 \not< 2$)

$n^2/1000 \ne o(n^2)$

O() versus o()

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$, for all $n \geq n_0\}$.

$o(g(n)) = \{f(n)$: for *any* positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.

Thus o*(f(n))* is a weakened O*(f(n))*.

For example: $n^2 = O(n^2)$

$$n^2 \neq o(n^2)$$

$$n^2 = O(n^3)$$

$$n^2 = o(n^3)$$

ω-notation

We say $g(n)$ is a *lower bound* for $f(n)$ that is not asymptotically tight.

$$\omega(g(n)) = \{f(n) : \text{for all constants } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\}.$$

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$n^{2.0001} = \omega(n^2)$

$n^2 \lg n = \omega(n^2)$

$n^2 \neq \omega(n^2)$

Properties

- Transitivity

$$f(n) = \Theta(g(n)) \;\&\; g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$
$$f(n) = O(g(n)) \;\&\; g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$
$$f(n) = \Omega(g(n)) \;\&\; g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$

- Symmetry

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$

- Transpose Symmetry

$f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$

Some Common Name for Complexity

$O(1)$	Constant time
$O(\log n)$	Logarithmic time
$O(\log^2 n)$	Log-squared time
$O(n)$	Linear time
$O(n^2)$	Quadratic time
$O(n^3)$	Cubic time
$O(n^i)$ for some i	Polynomial time
$O(2^n)$	Exponential time

Growth Rates of some Functions

$$O(\log n) < O(\log^2 n) < O(\sqrt{n}) < O(n)$$
$$< O(n \log n) < O(n \log^2 n) < O(n^{1.5}) < O(n^2)$$
$$< O(n^3) < O(n^4)$$

$$O(n^c) = O(2^{c \log n}) \text{ for any constant } c$$

Polynomial Functions

$$< O(n^{\log n}) = O(2^{\log^2 n})$$
$$< O(2^n) < O(3^n) < O(4^n)$$
$$< O(n!) < O(n^n)$$

Exponential Functions

A Survey of Common Running Times

THIS SECTION SLIDES BY KEVIN WAYNE.

Why it matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	4 sec
$n = 30$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	18 min	10^{25} years
$n = 50$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	11 min	36 years	very long
$n = 100$	$<$ 1 sec	$<$ 1 sec	$<$ 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	$<$ 1 sec	$<$ 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	$<$ 1 sec	$<$ 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	$<$ 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

Linear Time: O(n)

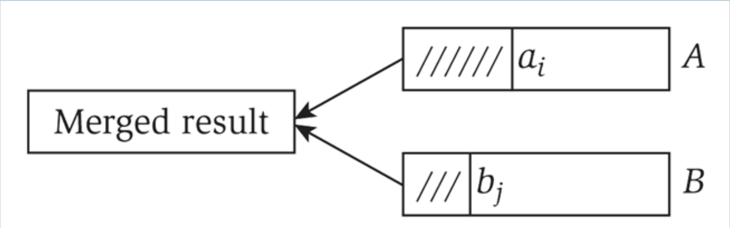
- Linear time. Running time is at most a constant factor times the size of the input.

- Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max = a1
for i = 2 to n {
   if (ai > max)
      max ← ai
}
```

Linear Time: O(n)

- Merge. Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole.




```
i = 1, j = 1
while (both lists are nonempty) {
   if (a_i ≤ b_j) append a_i to output list and increment i
   else(a_i > b_j)append b_j to output list and increment j
}
append remainder of nonempty list to output list
```

- Claim. Merging two lists of size n takes O(n) time.
- Pf. After each comparison, the length of output list increases by 1.

O(n lg n) Time

- O(n lg n) time. Arises in divide-and-conquer algorithms.

  also referred to as linearithmic time

- Sorting. Mergesort and heapsort are sorting algorithms that perform O(n lg n) comparisons.

- Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

- O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

- Quadratic time. Enumerate all pairs of elements.
- Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.
- $O(n^2)$ solution. Try all pairs of points.

```
min ← (x1 - x2)² + (y1 - y2)²
for i = 1 to n {
   for j = i+1 to n {
      d ← (xi - xj)² + (yi - yj)²      ← no need to
      if (d < min)                        take square roots
         min ← d
   }
}
```

- Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

Cubic Time: $O(n^3)$

- Cubic time. Enumerate all triples of elements.

- Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

- $O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set Si {
   foreach other set Sj {
      foreach element p of Si {
         determine whether p also belongs to Sj
      }
      if (no element of Si belongs to Sj)
         report that Si and Sj are disjoint
   }
}
```

Polynomial Time: $O(n^k)$ Time

- Independent set of size k. Given a graph, are there k nodes such that no two are joined by an edge?

  k is a constant

- $O(n^k)$ solution. Enumerate all subsets of k nodes.

```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

  - Check whether S is an independent set = $O(k^2)$.
  - Number of k element subsets = $\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k\,(k-1)\,(k-2)\cdots(2)\,(1)} \leq \frac{n^k}{k!}$
  - $O(k^2\, n^k / k!) = O(n^k)$.

  poly-time for k=17, but not practical

Exponential Time

- Independent set. Given a graph, what is maximum size of an independent set?

- $O(n^2 2^n)$ solution. Enumerate all subsets.

```
S* ← φ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```